Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(00)00127-1
J. Appl. Maths Mechs, Vol. 64, No. 6, pp. 959–965, 2000

0021–8928/00/$—see front matter

THE CONSTRUCTION OF DUAL VARIATIONAL PRINCIPLES OF THE THEORY OF THE SEEPAGE OF AN INCOMPRESSIBLE FLUID IN SOME DEFORMABLE COMPLEX MEDIA†

P. A. MAZUROV

Kazan

e-mail: mazurov@sci.kin.ru

(*Received* 27 *August* 1999)

A method of constructing dual variational principles for problems of the seepage of an incompressible fluid in deformable media with complex rheology is presented. Dual variational principles of the fluid seepage in a typical deformable media with complex rheology [1, 2], namely, a Maxwell element, connected in parallel with a visco-plastic or elastic element, are constructed. The variational principles are derived from variational problems compiled for the constitutive relations for fluid and solid phases.

The basis for compiling variational problems is a consideration of the minimum of the rate of energy dissipation and accumulation. One of the features of the construction of such variational principles is the coupling of the energy dissipation and accumulation mechanisms [3]. Another feature, unlike the methods described previously [4, 5], is the construction of dual variational principles that are independent of one another.

The purpose of the present paper is to develop further the method proposed in previous papers [6–9]‡ and to demonstrate its practicability when constructing new variational principles.

1. THE SCHEME FOR DERIVING THE VARIATIONAL PRINCIPLES

Consider a mechanical system, the behaviour of which is determined by N dissipative mechanisms [3] $\boldsymbol{\Psi}_i(\mathbf{Y}_i)$ ($i = 1, 2, \ldots, N$), where the dissipative potentials $\boldsymbol{\Psi}_i(\mathbf{Y}_i)$ are smooth convex functionals, $\mathbf{Y}_i = \mathbf{Y}_i(\mathbf{c})$ are generalized velocities and $\mathbf{c} = \{\mathbf{c}_\alpha\}$ is the set of independent variables $\mathbf{c}_\alpha$. For example, in the case of two dissipative mechanisms ($\mathbf{Y}_1 = \mathbf{e}(\dot{\mathbf{u}}), \mathbf{Y}_2 = \mathbf{q}$): $\mathbf{c} = \{\dot{\mathbf{u}}, \mathbf{q}\}$, where $\mathbf{e}$ is the deformation rate tensor, $\mathbf{q}$ is the seepage rate vector, $\dot{\mathbf{u}}$ is the displacement rate vector and $\mathbf{u}$ is the displacement vector. The dissipative mechanisms are assumed to be uncoupled [3], if any of the mechanisms $\boldsymbol{\Psi}_i(\mathbf{Y}_i)$ is independent of the variables $\mathbf{c}_\alpha$, on which the remaining dissipative mechanisms depend.

For steady processes, in the thermodynamics of irreversible processes, one of the variational principles is the principle of least energy dissipation, which is expressed by the minimum of the dissipative potentials [10, 11]

$$\int_\Omega \varphi(\cdot)d\Omega, \quad \varphi(\cdot) = \sum_{i=1}^N \boldsymbol{\Psi}_i(\mathbf{Y}_i) \tag{1.1}$$

We will assume that, for the mechanical system considered, under steady conditions, a variational principle exists, the basis of which is functional (1.1). The variational principle will then have the following form

$$\inf_{\mathbf{c}\in M_c} I_1(\mathbf{c}) = \inf_{\mathbf{c}\in M_c}\left[\int_\Omega (\varphi(\cdot) + f_1(\mathbf{c}))d\Omega + \int_\Gamma F_1(\mathbf{c})d\Gamma\right] \tag{1.2}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 6, pp. 996–1003, 2000.

‡See also: MAZUROV, P. A., A variational approach in theories of seepage consolidation and two-phase seepage. Kazan. Fiz.-Tekh. Inst., Kazan. Filial Akad. Nauk SSSR, Kazan, 1989. Deposited in VINITI 20.04.89, No. 2586-C.

where $f_1(\mathbf{c})$ and $F_1(\mathbf{c})$ are linear functionals of the variables $\mathbf{c}_\alpha$, and Γ is the boundary of the region Ω. It is clear from (1.2) that variational principles in other sets of variables can be written in the form

$$\inf_{\mathbf{c}\in M_c}\sup_{\mathbf{b}\in M_b} I(\mathbf{c},\mathbf{b}) = \inf_{\mathbf{c}\in M_c}\sup_{\mathbf{b}\in M_b}\left[\int_\Omega (\Psi(\cdot)-\Phi^*(\cdot)+f(\mathbf{c},\mathbf{b}))d\Omega + \int_\Gamma F(\mathbf{c},\mathbf{b})d\Gamma\right] \tag{1.3}$$

$$\Psi(\cdot)=\sum_{i=1}^{L}\Psi_i(\mathbf{Y}_i),\quad \Phi^*(\cdot)=\sum_{i=L+1}^{N}\Phi_i^*(\mathbf{X}_i),\quad \mathbf{X}_i=\operatorname{grad}\Psi_i(\mathbf{Y}_i)$$

where $\mathbf{X}_i = \mathbf{X}_i(\mathbf{b})$ are generalized forces, $\mathbf{b} = \{\mathbf{b}_\alpha\}$ is the set of independent variables $\mathbf{b}_\alpha$, $f(\mathbf{c}, \mathbf{b})$ and $F(\mathbf{c}, \mathbf{b})$ are linear functionals of the variables $\mathbf{c}_\alpha$ and $\mathbf{b}_\alpha$, and $\Phi_i^*(\mathbf{X}_i)$ are conjugate dissipative potentials, related to the dissipative potentials $\Psi_i(\mathbf{Y}_i)$ by a Young-Fenchel transformation [4]

$$\Phi_i^*(\mathbf{X}_i)=\sup_{\mathbf{Y}_i}[\mathbf{X}_i\mathbf{Y}_i-\Psi_i(\mathbf{Y}_i)]$$

Variational principles in the generalized velocities (1.2) and in the generalized forces

$$\inf_{\mathbf{b}\in M_b} I_2(\mathbf{b})=\inf_{\mathbf{b}\in M_b}\left[\int_\Omega(\varphi^*(\cdot)+f_2(\mathbf{b}))d\Omega+\int_\Gamma F_2(\mathbf{b})d\Gamma\right],\quad \varphi^*(\cdot)=\sum_{i=1}^{N}\Phi_i^*(\mathbf{X}_i)$$

are special forms of variational principle (1.3).

Note that, in general, for coupled dissipation mechanisms, the functional $\varphi^*(\cdot)$ will not be conjugate to the functional $\varphi(\cdot)$[3].

Suppose a solution and a variational principle (1.3) exists for a certain boundary-value problem. In variational principle (1.3) it is required to establish the form of the functionals $f(\mathbf{c}, \mathbf{b})$ and $F(\mathbf{c}, \mathbf{b})$, and also the set of constraints M_c and M_b, imposed on $\mathbf{c}_\alpha$ and b_α. We will introduce the following notation for the variables in the solution.

$$\mathbf{c}_\alpha=\mathbf{c}_\alpha^\circ,\quad \mathbf{b}_\alpha=\mathbf{b}_\alpha^\circ,\quad \mathbf{Y}_i=\mathbf{Y}_i(\mathbf{c}^\circ)=\mathbf{Y}_i^\circ,\quad \mathbf{X}_i=\mathbf{X}_i(\mathbf{b}^\circ)=\mathbf{X}_i^\circ$$

We will formulate variational principle (1.3) corresponding to the variational problem

$$\inf_{\mathbf{c}}\sup_{\mathbf{b}}\mathbf{B}(\mathbf{c},\mathbf{b})=\inf_{\mathbf{c}}\sup_{\mathbf{b}}\int_\Omega\left[\Psi(\cdot)-\Phi^*(\cdot)-\sum_{i=1}^{L}\mathbf{X}_i^\circ\mathbf{Y}_i+\sum_{i=L+1}^{N}\mathbf{X}_i\mathbf{Y}_i^\circ\right]d\Omega \tag{1.4}$$

The solution $\mathbf{c}_\alpha^\circ$, $\mathbf{b}_\alpha^\circ$ of problem (1.3) is also the solution of problem (1.4) [12, 13]. At the same time, even for a unique solution of problem (1.3), say $\mathbf{c}_\alpha^\circ$, $\mathbf{b}_\alpha^\circ$ problem (1.4) can have a set of other solutions $\mathbf{c}_\alpha$, $\mathbf{b}_\alpha$, for which $\mathbf{Y}_i(\mathbf{c}) = \mathbf{Y}_i^\circ$, $\mathbf{X}_i(\mathbf{b}) = \mathbf{X}_i^\circ$. To construct a valid variational principle (1.3) it is necessary to convert variational problem (1.4) to the form (1.3). The conversions are assumed to be acceptable if $\mathbf{c}_\alpha^\circ$, $\mathbf{b}_\alpha^\circ$ is a solution of the variational problems related by transformations.

We will assume that, in the system considered, in addition to dissipation there is also accumulation of elastic energy, defined by the elastic convex smooth potentials $W_j(\mathbf{Z}_j)$ $(j = 1, 2, \ldots, K)$. For example, $\mathbf{Z}_j = \boldsymbol{\epsilon}_j$ is the strain tensor and $\partial W_j(\mathbf{Z}_j)/\partial\mathbf{Z}_j = \mathbf{P}_j$, $\mathbf{P}_j = \boldsymbol{\sigma}_j$ is the stress tensor. By analogy with (1.1) we will write the functional which reflects the rate of energy dissipation and accumulation.

$$\int_\Omega\left[\sum_{i=1}^{N}\Psi_i(\mathbf{Y}_i)+\sum_{j=1}^{K}\dot W_j(\mathbf{Z}_j)\right]d\Omega \tag{1.5}$$

The derivatives $\dot W_j(\mathbf{Z}_j)$ in (1.5) can be represented in one of the following forms

$$\dot W_j(\mathbf{Z}_j)=\frac{\partial W_j(\mathbf{Z}_j)}{\partial\mathbf{Z}_j}\dot{\mathbf{Z}}_j\quad\text{or}\quad \dot W_j(\mathbf{Z}_j)\approx\frac{W_j(\mathbf{Z}_j)-W_j(\mathbf{Z}_j^{k-1})}{\Delta t} \tag{1.6}$$

where all the variables are given at the current instant of time t, with the exception of $\mathbf{Z}_j^{k-1} = \mathbf{Z}_j(t - \Delta t)$. In the first representation of (1.6) $\dot{\mathbf{Z}}_j = \dot{\mathbf{Z}}_j(\mathbf{c})$, while in the second $\mathbf{Z}_j = \mathbf{Z}_j(\mathbf{c})$. If functional (1.5) is convex in the variable $\mathbf{c}_\alpha$, the first representation for $\dot W_j(\mathbf{Z}_j)$ is possible if the variables $\mathbf{c}_\alpha$, occurring in

$\dot{\mathbf{Z}}_j(\mathbf{c})$, are expressed in terms of the variables $\mathbf{c}_\alpha$, occurring in $\Psi_i(\mathbf{Y}_i)$ and in those potentials $W_j(\mathbf{Z}_j)$ which are given by the second representation. In the case of two accumulation mechanisms

$$\dot{W}_1(\mathbf{Z}_1)=\frac{\partial W_1(\mathbf{Z}_1)}{\partial \mathbf{Z}_1}\dot{\mathbf{Z}}_1 \text{ and } \dot{W}_2(\mathbf{Z}_2)\approx\frac{W_2(\mathbf{Z}_2)-W_2(\mathbf{Z}_2^{k-1})}{\Delta t}$$

the variational principle, like (1.2), will have a similar form, where

$$\varphi(\cdot)=\sum_{i=1}^{N}\Psi_i(\mathbf{Y}_i)+\frac{\partial W_1(\mathbf{Z}_1)}{\partial \mathbf{Z}_1}\dot{\mathbf{Z}}_1+\frac{W_2(\mathbf{Z}_2)-W_2(\mathbf{Z}_2^{k-1})}{\Delta t}$$

while the corresponding variational problem will be

$$\inf_{\mathbf{c}}\mathbf{B}_1(\mathbf{c})=\inf_{\mathbf{c}}\int_{\Omega}\left[\varphi(\cdot)-\sum_{i=1}^{N}\mathbf{X}_i^{\circ}\mathbf{Y}_i-\mathbf{P}_1^{\circ}\mathbf{Z}_1-\mathbf{P}_2^{\circ}\frac{\mathbf{Z}_2-\mathbf{Z}_2^{k-1}}{\Delta t}\right]d\Omega$$

To construct dual variational principles we will write the variational problems similar to (1.4).

The above discussion holds for constitutive relations of the subgradient type [12, 13] $\mathbf{X}_i\in\partial\Psi_i(\mathbf{Y}_i)$ ($\mathbf{P}_j\in\partial W_j(\mathbf{Z}_j)$) where $\Psi_i(\mathbf{Y}_i)$ are convex eigenfunctionals, semicontinuous from below, $\mathbf{X}_i$ is the subgradient of the functional $\Psi_i(\mathbf{Y}_i)$ at the point $\mathbf{Y}_i$, and $\partial\Psi_i(\mathbf{Y}_i)$ is the set of all subgradients of the functional $\Psi_i(\mathbf{Y}_i)$ at the point $\mathbf{Y}_i$, consisting of one element grad $\Psi_i(\mathbf{Y}_i)$ in the case of smooth $\Psi_i(\mathbf{Y}_i)$.

2. CONSTRUCTION OF THE VARIATIONAL PRINCIPLE

We will write the system of equations of the seepage of an incompressible fluid in a deformable medium [14] in the form

$$\sigma^f_{ij,j}-p_{,i}=0\quad(\Pi_{ij,j}=0)\tag{2.1}$$

$$\operatorname{div}\mathbf{q}+\operatorname{div}\dot{\mathbf{u}}=0\tag{2.2}$$

$$\mathbf{q}=-\partial\Phi_p(\nabla p)/\partial\nabla p\ \text{ or }\ -\nabla p=\partial\Psi_q(\mathbf{q})/\partial\mathbf{q}\tag{2.3}$$

$$\sigma^f_{ij}=F_{ij}(\varepsilon_{ij},e_{ij})\tag{2.4}$$

where (2.1) and (2.2) are balancing equations, (2.3) and (2.4) are the constitutive relations for the fluid and solid phases, $\Psi_q(\mathbf{q})$, $\Phi_p(\nabla p)$ are the dissipative and conjugate dissipative potentials for the fluid phase, p is the pressure, u_i are the components of the displacement vector $\mathbf{u}$, σ^f_{ij} are the components of the effective stress tensor $\boldsymbol{\sigma}^f$, $\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$ are the components of the strain tensor $\boldsymbol{\epsilon}$, $e_{ij}=e_{ij}$, Π_{ij} are the components of the total stress tensor $\boldsymbol{\Pi}=\boldsymbol{\sigma}^f-\mathbf{p}$, and $\mathbf{p}$ is a spherical tensor with components $\delta_{ij}p$.

The solid phase is modelled by a Maxwell element, connected in parallel with a viscous element [1, 2]. In the Maxwell element the dissipative mechanism is determined by the potential $\Psi_1(e_1)$, the accumulation mechanism is determined by the potential $W_2(e_2)$, and in the viscous element the dissipative mechanism is determined by the potential $\Psi_3(\mathbf{e}_3)$. The constitutive relations (2.4) of this solid phase have the form

$$\boldsymbol{\sigma}_1=\partial\Psi_1(\mathbf{e}_1)/\partial\mathbf{e}_1,\quad\boldsymbol{\sigma}_2=\partial\Psi_2(\boldsymbol{\epsilon}_2)/\partial\boldsymbol{\epsilon}_2,\quad\boldsymbol{\sigma}_3=\partial\Psi_3(\mathbf{e}_3)\partial\mathbf{e}_3\tag{2.5}$$

$$\boldsymbol{\sigma}^f=\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_3,\quad\boldsymbol{\sigma}_1=\boldsymbol{\sigma}_2,\quad\mathbf{e}=\mathbf{e}_1+\mathbf{e}_2=\mathbf{e}_3\tag{2.6}$$

From considerations of the minimum rate of energy dissipation and accumulation, functionals (1.5) for the two representations of the potential $W_2(\boldsymbol{\epsilon}_2)$ will have the following respective forms

$$\int_{\Omega}\varphi_1(\cdot)d\Omega=\int_{\Omega}\left[\Psi_1(\mathbf{e}_1)+\frac{\partial W_2(\boldsymbol{\epsilon}_2)}{\partial\boldsymbol{\epsilon}_2}\mathbf{e}_2+\Psi_3(\mathbf{e}_3)+\Psi_q(\mathbf{q})\right]d\Omega\tag{2.7}$$

$$\int_\Omega \varphi_2(\cdot)d\Omega = \int_\Omega \left[\Psi_1(\mathbf{e}_1) + \frac{W_2(\boldsymbol{\epsilon}_2) - W_2(\boldsymbol{\epsilon}_2^{k-1})}{\Delta t} + \Psi_3(\mathbf{e}_3) + \Psi_q(\mathbf{q}) \right] d\Omega \tag{2.8}$$

Taking relations (2.6) and the equality $\boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_2^{k-1} + \mathbf{e}_2\Delta t$ into account for $W/_2(\boldsymbol{\epsilon}_2)$ in (2.8) we conclude that functionals (2.7) and (2.8) have three independent variables $\mathbf{c}_\alpha$. Variational problem (1.4) for functional (2.7) when $\mathbf{c} = \{\dot{\mathbf{u}}_1, \dot{\mathbf{u}}, \mathbf{q}\}$, can be written in the form

$$\inf_{\mathbf{c}} B_1(\mathbf{c}) = \inf_{\mathbf{c}} \int_\Omega \left[\varphi_1(\cdot) - \sum_{i=1}^{N} \overset{\circ}{\boldsymbol{\sigma}}_i \mathbf{e}_i + \nabla \overset{\circ}{p}\mathbf{q} \right] d\Omega \tag{2.9}$$

Taking relations (2.1)–(2.6) into account we convert problem (2.9) to a variational principle

$$\inf_{\mathbf{c}} B_1(\mathbf{c}) = \inf_{\mathbf{c}} \int_\Omega \left[\varphi_1(\cdot) - (\overset{\circ}{\boldsymbol{\sigma}}_1 \mathbf{e}_1 + \overset{\circ}{\boldsymbol{\sigma}}_2(\mathbf{e}_3 - \mathbf{e}_1) + \overset{\circ}{\boldsymbol{\sigma}}_3\mathbf{e}_3) + \nabla \overset{\circ}{p}\mathbf{q} \right] d\Omega =$$

$$= \inf_{\mathbf{c}} \int_\Omega \left[\varphi_1(\cdot) - \overset{\circ}{\boldsymbol{\sigma}}_f \mathbf{e} + \nabla \overset{\circ}{p}\mathbf{q} \right] d\Omega = \inf_{\mathbf{c}} \int_\Omega \left[\varphi_1(\cdot) - \overset{\circ}{\Pi}_{ij}\dot{u}_{ij} - \overset{\circ}{p}\dot{u}_{i,i} + \overset{\circ}{p}_{,i} q_i \right] d\Omega =$$

$$= \inf_{\mathbf{c}} \left[\int_\Omega \varphi_1(\cdot)d\Omega - \int_\Gamma \overset{\circ}{\Pi}_{ij} n_j \dot{u}_i d\Gamma + \int_\Gamma \overset{\circ}{p} q_n d\Gamma - \int_\Omega \overset{\circ}{p}(\dot{u}_{i,i} + q_{i,i}) d\Omega \right] =$$

$$= \inf_{\mathbf{c}\in(2.2)} \left[\int_\Omega (\Psi_1(\mathbf{e}_1) + \overset{\circ}{\boldsymbol{\sigma}}_2(\mathbf{e} - \mathbf{e}_1) + \Psi_3(\mathbf{e}) + \Psi_q(\mathbf{q})) d\Omega - \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_3 \right] =$$

$$= \inf_{\dot{\mathbf{u}}_1} \int_\Omega \left[\Psi_1(\mathbf{e}_1) - \overset{\circ}{\boldsymbol{\sigma}}_2 \mathbf{e}_1 \right] d\Omega + \inf_{\dot{\mathbf{u}},\mathbf{q}\in(2.2)} \left[\int_\Omega (\Psi_3(\mathbf{e}) + \overset{\circ}{\boldsymbol{\sigma}}_2\mathbf{e} + \Psi_q(\mathbf{q})) d\Omega - \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_3 \right] \tag{2.10}$$

$$\boldsymbol{\Gamma}_1 = \int_{\Gamma_1} \overset{\circ}{\Pi}_{ij} n_j \dot{u}_i d\Gamma, \quad \boldsymbol{\Gamma}_3 = \int_{\Gamma_3} \overset{\circ}{p} q_n d\Gamma, \quad \overset{\circ}{\boldsymbol{\sigma}}_2 = \frac{\partial W_2(\boldsymbol{\epsilon}_2)}{\partial \boldsymbol{\epsilon}_2}$$

with the conditions

$$\dot{u}_i = \overset{\circ}{\dot{u}}_i \text{ on } \Gamma_2, \quad q_n = \overset{\circ}{q}_n \text{ on } \Gamma_4, \Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4 = \Gamma \tag{2.11}$$

The expressions $\mathbf{c} \in (2.2)$ and $\dot{\mathbf{u}}, \mathbf{q} \in (2.2)$ in variational principle (2.10 denote that inf is considered on the set of these variables which satisfy balancing equation (2.2). Variational principle (2.10) can be split into two variational principles, which are independent at each instant of time

$$\inf_{\mathbf{e}_1} I_{11}(\mathbf{e}_1) = \inf_{\mathbf{e}_1} \int_\Omega (\Psi_1(\mathbf{e}_1) - \overset{\circ}{\boldsymbol{\sigma}}_2 \mathbf{e}_1) d\Omega \tag{2.12}$$

$$\inf_{\dot{\mathbf{u}},\mathbf{q}\in(2.2),(2.11)} I_{12}(\dot{\mathbf{u}},\mathbf{q}) = \inf_{\dot{\mathbf{u}},\mathbf{q}\in(2.2),(2.11)} \left[\int_\Omega (\Psi_3(\mathbf{e}) + \overset{\circ}{\boldsymbol{\sigma}}_2\mathbf{e} + \Psi_q(\mathbf{q})) d\Omega - \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_3 \right] \tag{2.13}$$

Variational principles (2.12) and (2.13) correspond to the following scheme of the numerical solution

$$\boldsymbol{\sigma}_2(t) \overset{(2.12)(2.13)}{\Rightarrow} \mathbf{e}_1(t), \dot{\mathbf{u}}(t), \mathbf{q}(t) \overset{\boldsymbol{\epsilon}_1(t),\boldsymbol{\epsilon}(t)}{\Rightarrow} \boldsymbol{\epsilon}_1(t+\Delta t), \boldsymbol{\epsilon}(t+\Delta t) \overset{(2.6)}{\Rightarrow} \boldsymbol{\epsilon}_2(t+\Delta t) \overset{(2.5)}{\Rightarrow} \boldsymbol{\sigma}_2(t+\Delta t)$$

From this scheme we understand the sufficiency of the definition of the variable $\mathbf{e}_1$ instead of $\dot{\mathbf{u}}_1$ for the numerical solution of the problem.

We will write variational problem (1.4) for functional (2.8) when $\mathbf{c} = \{\dot{\mathbf{u}}_1, \dot{\mathbf{u}}, \mathbf{q}\}$

$$\inf_{\mathbf{c}} B_2(\mathbf{c}) = \inf_{\mathbf{c}} \int_\Omega \left[\varphi_2(\cdot) - \overset{\circ}{\boldsymbol{\sigma}}_1\mathbf{e}_1 - \overset{\circ}{\boldsymbol{\sigma}}_2 \frac{\boldsymbol{\epsilon}_2 - \boldsymbol{\epsilon}_2^{k-1}}{\Delta t} - \overset{\circ}{\boldsymbol{\sigma}}_3\mathbf{e}_3 + \nabla \overset{\circ}{p}\mathbf{q} \right] d\Omega \tag{2.14}$$

Conversions of problem (2.14), similar to (2.10), taking into account the note regarding the variables $\dot{\mathbf{u}}_1$ and $\mathbf{e}_1$ lead to the following variational principle

$$\inf_{\mathbf{e}_1,\dot{\mathbf{u}},\mathbf{q}\in(2.2),(2.11)} I_2(\mathbf{e}_1,\dot{\mathbf{u}},\mathbf{q}) =$$

$$= \inf_{\mathbf{e}_1,\dot{\mathbf{u}},\mathbf{q}\in(2.2),(2.11)}\left[\int_\Omega\left(\Psi_1(\mathbf{e}_1)+\frac{W_2(\boldsymbol{\epsilon}_2^{k-1}+(\mathbf{e}-\mathbf{e}_1)\Delta t)-W_2(\boldsymbol{\epsilon}_2^{k-1})}{\Delta t}+\Psi_3(\mathbf{e})+\Psi_q(\mathbf{q})\right)d\Omega-\right.$$

$$\left.-\boldsymbol{\Gamma}_1+\boldsymbol{\Gamma}_3\right] \tag{2.15}$$

Variational principle (2.15) corresponds to the numerical scheme

$$\boldsymbol{\epsilon}_2(t)\overset{(2.15)}{\Rightarrow}\mathbf{e}_1(t+\Delta t),\quad \dot{\mathbf{u}}(t+\Delta t),\quad \mathbf{q}(t+\Delta t)\overset{\mathbf{e}_1(t),\mathbf{e}(t)}{\Rightarrow}\boldsymbol{\epsilon}_1(t+\Delta t),\boldsymbol{\epsilon}(t+\Delta t)\overset{(2.6)}{\Rightarrow}\boldsymbol{\epsilon}_2(t+\Delta t)$$

It can be seen from the variational principles constructed that the minimum of the functionals (2.7) and (2.8), which characterize the rate of energy dissipation and accumulation, is reached on the solution of the initial problem when appropriate boundary conditions are specified for the variables $\mathbf{c}_\alpha$ over the whole boundary Γ ($\Gamma_1=\varnothing, \Gamma_3=\varnothing$) of the domain of the solution Ω and when these variables satisfy the conservation conditions (2.2) and the constraints (2.6) imposed. It is of interest to use similar considerations to derive the constitutive relations and relations between the thermodynamic forces in media with complex rheology from the condition for functionals of the form (1.5) to be a minimum. This approach was used when deriving the constitutive relations obtained previously from dimensionality considerations [15], for the fluid seepage in media with double porosity [8].

We will construct a variational principle for the set of variables $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3, p$. The basis of the variational principle for this set is the functional

$$\int_\Omega \varphi_3(\cdot)d\Omega=\int_\Omega\left[\Phi_1(\boldsymbol{\sigma}_1)+\frac{W_2^*(\boldsymbol{\sigma}_2)-W_2^*(\boldsymbol{\sigma}_2^{k-1})}{\Delta t}+\Phi_3(\boldsymbol{\sigma}_3)+\Phi_p(\nabla p)\right]d\Omega$$

Taking relations (2.6) into account we choose the independent variables $\mathbf{b}=\{\boldsymbol{\sigma}_1,\boldsymbol{\Pi},p\}$ and we write the initial variational problem

$$\inf_{\mathbf{b}} B_3(\mathbf{b})=\inf_{\mathbf{b}}\int_\Omega\left[\varphi_3(\cdot)-\mathring{\mathbf{e}}_1\boldsymbol{\sigma}_1-\frac{\mathring{\boldsymbol{\epsilon}}_2}{\Delta t}\boldsymbol{\sigma}_2-\mathring{\mathbf{e}}_3\boldsymbol{\sigma}_3+\mathbf{q}^\circ\nabla p\right]d\Omega \tag{2.16}$$

We convert problem (2.16) to the variational principle

$$\inf_{\mathbf{b}} B_3(\mathbf{b})=$$

$$=\inf_{\mathbf{b}}\int_\Omega\left[\varphi_3(\cdot)-\mathring{\mathbf{e}}_1\boldsymbol{\sigma}_1-\frac{\mathring{\boldsymbol{\epsilon}}_2-\boldsymbol{\epsilon}_2^{k-1}}{\Delta t}\boldsymbol{\sigma}_1-\frac{\boldsymbol{\epsilon}_2^{k-1}}{\Delta t}\boldsymbol{\sigma}_1-\mathbf{e}^\circ(\boldsymbol{\sigma}^f-\boldsymbol{\sigma}_1)+\mathbf{q}^\circ\nabla p\right]d\Omega=$$

$$=\inf_{\mathbf{b}}\int_\Omega\left[\varphi_3(\cdot)-\frac{\boldsymbol{\epsilon}_2^{k-1}}{\Delta t}\boldsymbol{\sigma}_1-\mathring{\dot u}_{i,1}(\Pi_{ij}+\delta_{ij}p)+\mathring{q}_i p_{,i}\right]d\Omega=$$

$$=\inf_{\mathbf{b}}\left[\int_\Omega\left(\varphi_3(\cdot)-\frac{\boldsymbol{\epsilon}_2^{k-1}}{\Delta t}\boldsymbol{\sigma}_1+\dot u_i^\circ\Pi_{ij,j}-(\mathring{\dot u}_{i,i}+\mathring{q}_{i,i})p\right)d\Omega-\right.$$

$$\left.-\int_\Gamma\Pi_{ij}n_j\mathring{\dot u}_i d\Gamma+\int_\Gamma\mathring{q}_n p d\Gamma\right]=\inf_{\mathbf{b}\in(2.1)}\left[\int_\Omega\left(\varphi_3(\cdot)-\frac{\boldsymbol{\epsilon}_2^{k-1}}{\Delta t}\boldsymbol{\sigma}_1\right)d\Omega-\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_4\right]$$

$$\boldsymbol{\Gamma}_2=\int_{\Gamma_2}\Pi_{ij}n_j\mathring{\dot u}_i d\Gamma,\quad \boldsymbol{\Gamma}_4=\int_{\Gamma_4}\mathring{q}_n p d\Gamma$$

with the conditions

$$\Pi_{ij}n_j=\mathring{F}_i \text{ on } \Gamma_1,\ p=p^\circ \text{ on } \Gamma_3 \tag{2.17}$$

Hence, we obtain the variational principle

$$\inf_{\sigma_1,\Pi,p\in(2.1),(2.17)} I_3(\sigma_1,\Pi,p)=\inf_{\sigma_1,\Pi,p\in(2.1),(2.17)}\left[\int_\Omega\left(\Phi_1(\sigma_1)+\frac{W_2^*(\sigma_1)-\epsilon_2^{k-1}\sigma_1}{\Delta t}+\right.\right.$$
$$\left.+\Phi_3(\Pi+\mathbf{p}-\sigma_1)+\Phi_p(\nabla p)\right)d\Omega-\Gamma_2+\Gamma_4]$$

In a similar way one can construct binary variational principles for all sets of variables. We will write, without derivation, a few of the constructed binary variational principles.

The variational principle for the set σ_1, σ_2, $\mathbf{e}_3, p$ (we take the independent variables σ_1, $\dot{\mathbf{u}}, p$) has the form

$$\inf_{\dot{\mathbf{u}}}\sup_{\sigma_1,p} I_4(\dot{\mathbf{u}},\sigma_1,p)=\inf_{\dot{\mathbf{u}}}\sup_{\sigma_1,p}\left[\int_\Omega(-\Phi_1(\sigma_1)-\frac{W_2^*(\sigma_1)-\epsilon_2^{k-1}\sigma_1}{\Delta t}+\Psi_3(\mathbf{e})-\right.$$
$$-\Phi_p(\nabla p)+\sigma_1\mathbf{e}-p\,\mathrm{div}\,\dot{\mathbf{u}})d\Omega-\Gamma_1-\Gamma_4]$$

with the conditions

$$\dot{u}_i=\overset{\circ}{u}_i \text{ on } \Gamma_2,\quad p=p^\circ \text{ on } \Gamma_3$$

The variational principle for the set of variables $\mathbf{e}_1$, σ_2, δ_3, p (we take the independent variables $\mathbf{u}_1$, δ_2, Π) has the form

$$\inf_{\mathbf{e}_1}\sup_{\sigma_2,\Pi,p\in(2.1),(2.17)} I_5(\mathbf{e}_1,\sigma_2,\Pi,p)=\inf_{\mathbf{e}_1}\sup_{\sigma_2,\Pi,p\in(2.1),(2.17)}\left[\int_\Omega(\Psi_1(\mathbf{e}_1)-\frac{W_2^*(\sigma_2)-\epsilon_2^{k-1}\sigma_2}{\Delta t}-\right.$$
$$-\Phi_3(\Pi+\mathbf{p}-\sigma_2)-\Phi_p(\nabla p)-\sigma_2\mathbf{e}_1)d\Omega+\Gamma_2-\Gamma_4]$$

The variational principle for the set of variables $\mathbf{e}_1$, σ_2, σ_3, $\mathbf{q}$ (we take the independent variables $\dot{\mathbf{u}}$, σ_2, Π, $\mathbf{q}$) has the form

$$\inf_{\mathbf{e}_1,\mathbf{q}}\sup_{\sigma_2,\Pi,p\in(2.1)} I_6(\mathbf{e}_1,\sigma_2,\Pi,\mathbf{q},p)=\inf_{\mathbf{e}_1,\mathbf{q}}\sup_{\sigma_2,\Pi,p\in(2.1)}\left[\int_\Omega(\Psi_1(\mathbf{e}_1)-\frac{W_2(\sigma_2)-\epsilon_2^{k-1}\sigma_2}{\Delta t}-\right.$$
$$-\Phi_3(\Pi+\mathbf{p}-\sigma_2)+\Psi_q(\mathbf{q})-\sigma_2\mathbf{e}_1-p\,\mathrm{div}\,\mathbf{q})d\Omega+\Gamma_2+\Gamma_3]$$

with the conditions

$$\Pi_{ij}n_j=\overset{\circ}{F}_i,\quad \text{on } \Gamma_1, q_n=\overset{\circ}{q}_n \text{ on } \Gamma_4$$

Note that, when deriving this variational principle, there is an additional variable p.

One can similarly construct binary variational principles when modelling the solid phase by a Maxwell element, connected in series with an elastic Maxwell element, connected in series with an elastic element $\sigma_3=\partial W_3(\epsilon_3)/\partial\epsilon_3$. One of the variational principles constructed for the set $\mathbf{e}_1$, ϵ_2, ϵ_3, p (we take the independent variables $\dot{\mathbf{u}}_1$, u, p) has the form

$$\inf_{\mathbf{e}_1,\mathbf{u}}\sup_{p} I_7(\mathbf{e}_1,\mathbf{u},p)=\inf_{\mathbf{e}_1,\mathbf{u}}\sup_{p}\left[\int_\Omega(\Psi_1(\mathbf{e}_1)+\right.$$
$$+\frac{W_2(\epsilon-\epsilon_1^{k-1}-\mathbf{e}_1\Delta t)-W_2(\epsilon_2^{k-1})}{\Delta t}+\frac{W_3(\epsilon)-W_3(\epsilon^{k-1})}{\Delta t}-$$
$$\left.-\Phi_p(\nabla p)-p\,\mathrm{div}\left(\frac{\mathbf{u}-\mathbf{u}^{k-1}}{\Delta t}\right)\right)d\Omega-\Gamma_1-\Gamma_4\Bigg],\quad \Gamma_1=\int_{\Gamma_1}\overset{\circ}{\Pi}_{ij}n_j\frac{u_i-u_i^{k-1}}{\Delta t}d\Gamma$$

with the conditions

$$u_i=\overset{\circ}{u}_i \text{ on } \Gamma_2, p=p^\circ \text{ on } \Gamma_3$$

The quantities $W_2(\epsilon_2^{k-1})$, $W_3(\epsilon^{k-1})$, u_i^{k-1} are written for physical clarity and may be omitted as constants.

A direct check shows that if the variations of the functionals $I_i(\cdot)$ are zero, this is equivalent to system (2.1–(2.6) with specified boundary conditions. The variational principles constructed hold when the potentials $\Psi_i(\cdot)$ are non-differentiable and represent motion of viscoplastic and rigidly plastic bodies [16]. The proposed method of constructing variational principles does not require a knowledge of the boundary conditions. The boundary conditions required for the solution are determined when converting the variational problems.

Note that individual variational principles [17–21],† obtained by different methods, have been derived using the proposed scheme.

3. CONCLUSION

Using the proposed approach, one can construct many new variational principles for well-known complex media or combinations of them, but this becomes a matter of technique and does not contain sufficient scientific novelty without applying variational principles to the solution of specific problems or without taking other features into account. The problems involved in obtaining constitutive relations (from considerations of the minimum rate of energy dissipation and accumulation) and the boundary conditions for new media with complex rheology remain of interest. In this case an analysis of a complex medium begins with an investigation of the energy dissipation and accumulation mechanisms, and also the relations and constraints to which the parameters occurring in the potentials are subject.

I wish to thank my teacher A. V. Kosterin.

REFERENCES

1. REINER, M. *Rheology*. Springer, Berlin, 1958.
2. IVLEV, D. D. and BYKOVTSEV, G. I., *Theory of Reinforcing Plastic Solid*. Nauka, Moscow, 1971.
3. GERMAIN, P., Cours de mecanique des milieux continus. Masson, Paris, 1973.
4. BERDICHEVSKII, V. L., *Variational Principles of Continuum Mechanics*. Nauka, Moscow, 1983.
5. EKLAND, I. and TEMAM, R., *Convex Analysis and Variational Problems*. North-Holland, Amsterdam, 1976.
6. MAZUROV, P. A., A variational approach in theories of seepage consolidation and two-phase seepage. *Prikl. Mat. Mekh.*, 1992, **56**, 1, 77–86.
7. MAZUROV, P. A., The construction of variational principles of two-phase seepage in adeformed medium. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1992, **5**, 1, 76–82.
8. MAZUROV, P. A., Variational principles of the seepage of an incompressible fluid in a media with double porosity. *Prikl. Mat. Mekh.*, 1993, **57**, 1, 65–70.
9. MAZUROV, P. A., The construction of variational principles. *Prikl. Mat. Mekh.*, 1997, **61**, 3, 432–439.
10. BAZAROV, I. P., *Thermodynamics*. Vyssh. Shkola, Moscow, 1983.
11. BAZAROV, I. P., GEVORKYAN, E. V. and NIKOLAYEV, P. N., *Non-equilibrium Thermodynamics and Physical Kinetics*. Izd. MGU, Moscow, 1989.
12. ROCKAFELLAR, R. T., *Convex Analysis*. University Press, Princeton, 1970.
13. PANAGIOTOPOULOS, P. D., *Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions*. Birkhauser, Boston, 1985.
14. NIKOLAYEVSKII, V. N., *The Mechanics of Porous and Cracked Media*. Nedra, Moscow, 1984.
15. BARENBLATT, G. I., ENTOV, V. M. and RYZHIK, V. M., *The Motion of Liquids and Gases in Natural Strata*. Nedra, Moscow, 1984.
16. MOSOLOV, P. P. and MYASNIKOV, V. P., *The Mechanics of Rigid Plastic Media*. Nauka, Moscow, 1981.
17. SANDHU, R. S. and WILSON, E. L., Finite element analysis of seepage in elastic media. *J. Eng. Meth. Div., ASCE*, 1969, **95**, 641–652.
18. SANDHU, R. S. and LIU, H., Analysis of consolidation of viscoelastic soils. *Numer. Meth. Geomech.: Proc. 3rd. Intn. Conf. Aachen*, 1979, Vol. 4. Rotterdam, 1980, 1255–1263.
19. NISSEKI, S. and SATAKE, M. Variational principles for nonlinear consolidation problem. *Numer. Mech. Geomech.: Proc. 3rd. Intn. Conf. Aachen*, 1979, Vol. 1. Rotterdam, 175–180.
20. WASHIZU, K., *Variational Methods in Elasticity and Plasticity*. Pergamon Press, Oxford, 1982.
21. GOL'DSHTEIN, R. V. and ENTOV, V. M., *Qualitative Methods in Continuum Mechanics*. Nauka, Moscow, 1989.

Translated by R.C.G.

†See also: KOSTERIN, A. V., A variational principle of seepage consolidation. Kazan. Univ., Kazan, 1986. Deposited at VINITI 16.12.86, No. 8598-V.